Examination Number:

Set:

Shore

Year 12

HSC Assessment Task 5 - Trial HSC

17th August 2012

Mathematics Extension 1

General Instructions

- Reading time − 5 minutes
- Working time − 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this question paper
- Answer Questions 1–10 on the Multiple Choice Answer Sheet provided
- Start each of Questions 11–14 in a new writing booklet
- Show all necessary working in Questions 11–14
- Write your examination number on the front cover of each booklet
- If you do not attempt a question, submit a blank booklet marked with your examination number and "N/A" on the front cover

Total marks – 70

Section I Pages 3–6

10 marks

- Attempt Questions 1−10
- Allow about 15 minutes for this section

Section II Pages 7–12

60 marks

- Attempt Questions 11−14
- Allow about 1 hour 45 minutes for this section

Note: Any time you have remaining should be spent revising your answers.

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the Multiple Choice Answer Sheet for Questions 1–10.

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1 The point P divides the interval AB, where $A = (-8, -2)$ and $B = (16, 10)$, internally in the ratio 1 : 3. What are the coordinates of P?

(A) $(-2, 1)$

(B) $(0, 2)$

(C) $(8, 6)$

(D) $(10, 7)$

2 The equation $x^3 - 5x + 2 = 0$ has roots α, β and 2. What is the value of $\alpha + \beta$?

(A) -7

(B) -2

(C) 2

(D) 3

3 What is the coefficient of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^5$?

(A) -160

(B) -80

(C) -32

(D) -2

4 What is the exact value of $\int_{-4}^{4} \frac{dx}{x^2 + 16}$?

(A) 0

(B) $\frac{\pi}{16}$

(C) $\frac{\pi}{8}$

(D) $2\ln 32$

5 Suppose $x^3 - (a+1)x + a \equiv (x+3)Q(x)$ where $Q(x)$ is a polynomial. Find the value of a.

(A) –6

(B) –3

(C) 3

(D) 6

6 The curves $y = x^2$ and $y = x^3$ intersect at the point (1, 1). Which of the following is closest to the size in radians of the acute angle between these curves at (1, 1)?

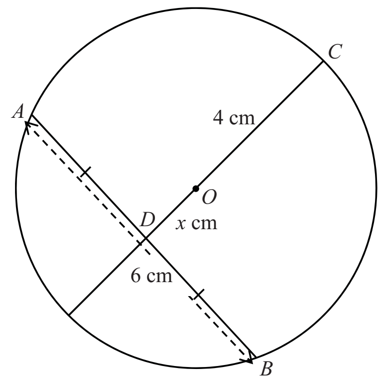
(A) 0

(B) 0.14

(C) 0.62

(D) $\frac{\pi}{4}$

7 In the diagram below the length of chord AB is 6 cm, $AD = BD$, the length of the radius OC is 4 cm and the length of OD is x cm.



NOT TO SCALE

What is the value of x?

(A) 2 cm

(B) $\sqrt{5}$ cm

(C) $\sqrt{7}$ cm

(D) 5 cm

8 It is known that $x^3 + 3x - 7 = 0$ has a root between $x = 1$ and $x = 2$. If the method of halving the interval is used twice, between which two values does the root lie?

(A) $x = 1$ and $x = 1.25$

(B) $x = 1.25$ and $x = 1.5$

(C) $x = 1.5$ and $x = 1.75$

(D) $x = 1.75$ and $x = 2$

9 Which one of these functions has an inverse relation that is not a function?

(A) $y = x^3$

(B) $y = \ln x$

(C) $y = \sqrt{x}$

(D) $y = |x|$

10 Which one of the following is the general solution of $2\sin^2\left(6t + \frac{\pi}{4}\right) = 1$?

(A) $t = \frac{n\pi}{12}$, where n is an integer.

(B) $t = \frac{n\pi}{12} - \frac{\pi}{24}$, where n is an integer.

(C) $t = \frac{n\pi}{3}$ and $t = \frac{n\pi}{3} + \frac{\pi}{12}$, where n is an integer.

(D) $t = \frac{n\pi}{3} - \frac{\pi}{6}$ and $t = \frac{n\pi}{3} + \frac{\pi}{12}$, where n is an integer.

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour 45 minutes for this section

Start each of Questions 11–14 in a new writing booklet.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Find the exact value of $\cos\left(\sin^{-1}\frac{4}{7}\right)$. **2**

(b) (i) Graph $y = |3x - 4|$. **1**

(ii) Hence, or otherwise, solve $|x - 6| \le |3x - 4|$. **2**

(c) Find $\frac{d}{dx}\left(x\cos^{-1}2x\right)$. **2**

(d) Use the substitution $u = x^3 + 1$ to find $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$. **3**

(e) By using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, show that **2**

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}.$$

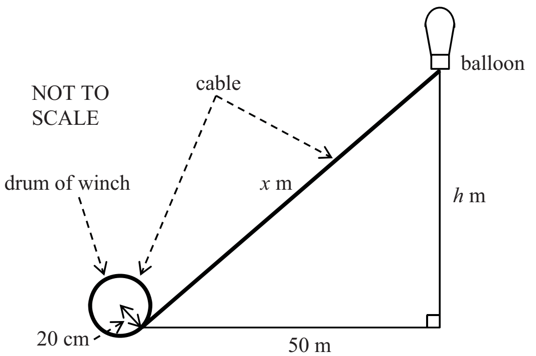
(f) Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = 3\cos\frac{x}{4}$, the x-axis and the lines $x = 0$ and $x = \pi$ is rotated about the x-axis. **3**

Question 12 **(15 marks)** Use a SEPARATE writing booklet **Marks**

(a) (i) How many nine-letter arrangements of the letters in the word SKEDADDLE are possible? **1**

(ii) How many nine-letter arrangements of the letters in the word SKEDADDLE are possible if the letters S and K must be adjacent? **1**

(b) A balloon ride consists of a hot-air balloon tethered to a winch by a strong thin cable. The drum of the winch has a radius of 20 cm. The operator of the ride must keep the balloon directly above its landing area, which is 50 m downwind of the winch. Let h metres be the height of the hot-air balloon and x metres be the length of the cable between the balloon and where it meets the winch.



(i) Find an expression for x in terms of h. **1**

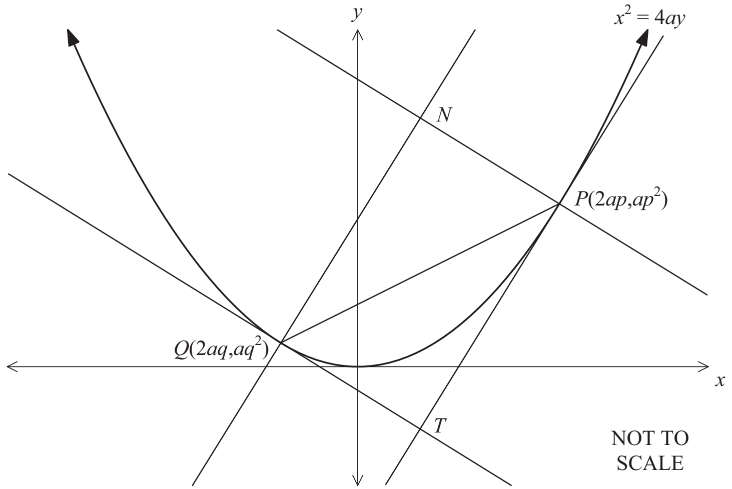
(ii) The winch operator must turn the winch so that the balloon rises vertically at a rate of 0.5 metres per second. **3**

At what rate $\dfrac{d\theta}{dt}$ must the winch operator turn the winch when the balloon is at a height of 20 metres?

(c) Sketch the graph of the function $f(x) = 3\cos^{-1}(2x-1)$. **2**
Clearly indicate the domain and range of the function.

Question 12 continues on the following page

(d) The diagram shows the parabola $x^2 = 4ay$. The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola such that PQ is a focal chord. The tangents at P and Q intersect at T and the normals at P and Q intersect at N.



(i) Show that the equation of any chord PQ is given by **2**

$$y = \left(\frac{p+q}{2}\right)x - apq.$$

(ii) Show that $pq = -1$ when PQ is a focal chord. **1**

(iii) Explain why $PTQN$ is a cyclic quadrilateral. **1**

(iv) Let C be the centre of the circle $PTQN$. **1**
Explain why C is the midpoint of PQ.

(v) Find the Cartesian equation of the locus of C. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Use mathematical induction to prove that for all integers $n \geq 1$, $11^{n+1} + 12^{2n-1}$ is divisible by 133. **3**

(b) A particle moves in a straight line along the x-axis so that its acceleration is given by $\ddot{x} = x + 3$ where x is the displacement from the origin.

Initially the particle is at the origin and has velocity $v = 3$.

(i) Show that $v = x + 3$. **2**

(ii) Find x as a function of t. **2**

(c) A multiple choice test consists of 30 questions, each having 4 possible answers. Ferdinand decides to answer each question by randomly choosing one of the four possible answers.

(i) What is the probability that Ferdinand will answer exactly half of the questions correctly? **1**

(ii) Find the most likely number of questions that Ferdinand answers correctly and the probability that he answers this many questions correctly. Give your answer correct to three significant figures. **3**

(d) A hot frying pan is cooling in a room of constant temperature 20°C. At time t minutes its temperature T decreases according to the equation

$$\frac{dT}{dt} = -k\left(T - 20\right) \text{ where } k \text{ is a positive constant.}$$

The initial temperature of the frying pan is 160°C and it cools to 100°C after 10 minutes.

(i) Verify that $T = 20 + Ae^{-kt}$ is a solution of this equation, where A is a constant. **1**

(ii) Find the values of A and k. **2**

(iii) How long will it take for the pan to become cool enough to touch (50°C)? Give your answer correct to the nearest minute. **1**

Question 14 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) A particle moves in a straight line. Its displacement, x metres, after t seconds is given by

$$x = \frac{2}{\sqrt{3}}\cos\left(3t\right) + 2\sin\left(3t\right).$$

(i) Show that the particle is moving in simple harmonic motion by showing that $\ddot{x} = -n^2 x$. **2**

(ii) By expressing x in the form $R\sin(3t + \alpha)$, with $0 < \alpha < \frac{\pi}{2}$, find the period and amplitude of the motion. **2**

(b) The binomial theorem states that **3**

$$(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k .$$

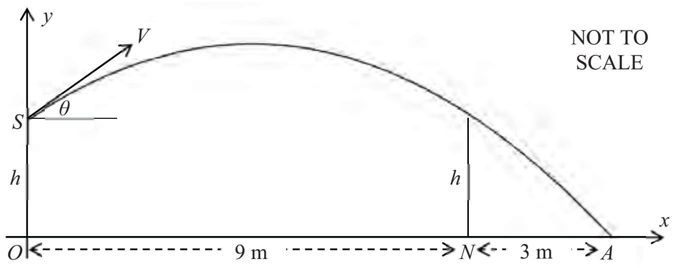
By integrating both sides of this identity with respect to x, show that

$$\frac{2^{n+1} - 1}{n + 1} = \sum_{k=0}^{n} \frac{1}{k + 1}\binom{n}{k}.$$

Question 14 continues on the following page

(c) In a game of volleyball the server, S, stands 9 metres from the net, N, and attempts to land a serve on the attack line, A, which is 3 metres from the net on the opposite side of the court.

At time $t = 0$, the ball is hit from S at a speed of V metres per second and an angle of elevation θ. Both the release point of the serve and the height of the net are h metres above the surface of the court. The point directly below S on the surface of the court should be taken as the origin, O, of the coordinate system.



The horizontal displacement of the ball at time t is given by $x = Vt\cos\theta$. (Do NOT show this).

(i) The equation of motion of the ball in the vertical direction is $\ddot{y} = -g$. Using calculus, show that the vertical displacement of the ball at time t is given by **2**

$$y = Vt\sin\theta - \tfrac{1}{2}gt^2 + h.$$

(ii) Hence show that the trajectory of the ball is given by **1**

$$y = h + x\tan\theta - \frac{g}{2V^2\cos^2\theta}x^2.$$

(iii) Show that when the ball clears the net **2**

$$V^2 > \frac{9g}{2\sin\theta\cos\theta}.$$

(iv) After clearing the net the ball hits the attack line A. Show that **2**

$$\tan\theta > \frac{h}{4}.$$

(v) If $h = 2.43$, find the minimum value of θ at which the ball can be served so that it clears the net and hits the attack line. **1**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

Note $\ln x = \log_e x, \quad x > 0$

Section 1

1. A
2. B
3. B
4. C
5. D
6. B
7. C
8. B
9. D
10. A

Section 1 - Worked

1. $P = \left(\frac{3\times-8+1\times16}{1+3}, \frac{3\times-2+1\times10}{1+3}\right)$
$= (-2, 1)$

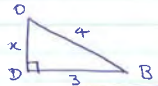
2. $\alpha+\beta+2=0$
$\therefore \alpha+\beta=-2$

3. $T_{k+1} = \binom{5}{k} x^{2(5-k)} \left(\frac{-2}{x}\right)^k$
$= \binom{5}{k} x^{10-2k} (-2)^k (x^{-k})$
$\therefore 10-3k = 1$
$k=3$
$\therefore$ coefficient of x
$= \binom{5}{3} \times (-2)^3$
$= -80$

4. $\int_{-4}^{4} \frac{dx}{x^2+4} = \left[\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right)\right]_{-4}^{4}$
$= \frac{1}{4}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right)$
$= \frac{\pi}{8}.$

5. $P(x) = x^3 - (a+1)x + a$
$P(-3) = 0$
$-27 + 3(a+1) + a = 0$
$4a + 3 = 27$
$a = 6$

6. $y = x^2$ at $(1,1) \Rightarrow m_1 = 2$
$y = x^3$ at $(1,1) \Rightarrow m_2 = 3$
$\therefore \tan\theta = \left|\frac{2-3}{1+2\times3}\right|$
$\theta = \tan^{-1}\left(\frac{1}{7}\right)$
$\doteqdot 0.14$

7. 

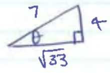
$x = \sqrt{4^2-3^2}$
$= \sqrt{7}$

8. $f(1) = -3$, $f(2) = 7$
$f(1.5) = \frac{7}{8}$, $f(1.25) < 0$.
$\therefore 1.25 < x \leq 1.5$

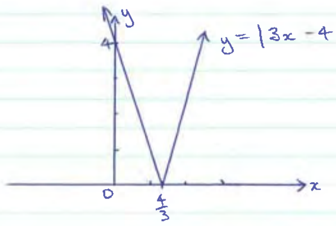
9. $y = |x|$

10. $\sin\left(6t+\frac{\pi}{4}\right) = \pm\frac{1}{\sqrt{2}}$
$6t + \frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}, \ldots$
$6t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, \ldots$
$t = 0, \frac{\pi}{12}, \frac{2\pi}{12}, \frac{3\pi}{12}, \frac{4\pi}{12}, \ldots$
$= \frac{n\pi}{12}, \; n \in \mathbb{Z}.$

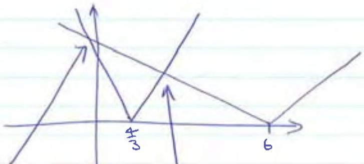
Section 2 - Question 11.

a) 

$\therefore \cos\left(\sin^{-1}\left(\frac{4}{7}\right)\right) = \frac{\sqrt{33}}{7}$

b) (i) 

(ii) Graphically,



$6 - x = 4 - 3x$
$2x = -2$
$x = -1$

$6 - x = 3x - 4$
$10 = 4x$
$x = \frac{5}{2}$

$\therefore x \leq -1$ or $x \geq \frac{5}{2}.$

Question 11

(c) $\frac{d}{dx}(x\cos^{-1}2x) = \frac{d(x)}{dx}\cdot\cos^{-1}2x + x\cdot\frac{d(\cos^{-1}2x)}{dx}$
$= \cos^{-1}2x - \frac{x}{\sqrt{\frac{1}{4}-x^2}}$
or $= \cos^{-1}2x - \frac{2x}{\sqrt{1-4x^2}}$

d) $\int \frac{x^2}{\sqrt{x^3+1}}\,dx$ — let $u = x^3+1$, $du = 3x^2\,dx$
$= \frac{1}{3}\int \frac{1}{\sqrt{u}}\,du$
$= \frac{1}{3}\cdot 2u^{\frac{1}{2}} + C$
$= \frac{2}{3}\sqrt{x^3+1} + C$

e) LHS $= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$
$= \sqrt{\frac{1-\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}}}$
$= \sqrt{\frac{1+t^2-2t}{1+t^2+2t}}$
$= \sqrt{\frac{(1-t)^2}{(1+t)^2}}$
$= \frac{1-t}{1+t}$
$= \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$
$=$ RHS.

Question 11

f) $V = \pi\int_0^{\pi}\left(3\cos\frac{x}{4}\right)^2 dx$
$= 9\pi\int_0^{\pi}\cos^2\left(\frac{x}{4}\right)dx$
$= 9\pi\int_0^{\pi}\frac{\cos\frac{x}{2}}{2} + \frac{1}{2}\,dx$
$= 9\pi\left[\sin\left(\frac{x}{2}\right) + \frac{x}{2}\right]_0^{\pi}$
$= 9\pi\left(1 + \frac{\pi}{2} - 0 - 0\right)$
$= 9\pi\left(1 + \frac{\pi}{2}\right)$

Question 12

a) (i) arrangements $= \frac{9!}{3!\,2!}$
$= 30240.$

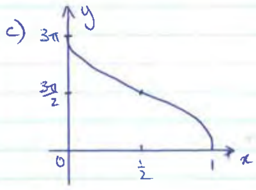
(ii) arrangements $= 2\times 8\times\frac{7!}{3!\,2!}$
$= 6720.$

b) (i) $x^2 = h^2 + 50^2$
$x = \sqrt{h^2+2500}$

(ii) $\frac{d\theta}{dt} = \frac{d\theta}{dl}\times\frac{dl}{dx}\times\frac{dx}{dh}\times\frac{dh}{dt}$
$= 5\times1\times\frac{h}{\sqrt{h^2+2500}}\times0.5$
$= \frac{2.5\times20}{\sqrt{2900}}$
$\doteqdot 0.93$ radians per second.

$l = r\theta$
$\frac{dl}{d\theta} = r$
$\frac{d\theta}{dl} = \frac{1}{r}$
$= \frac{1}{0.2}$
$= 5$
$\frac{dl}{dx} = 1$

c)



$0 \le x \le 1$ Domain

$0 \le y \le 3\pi$ Range.

d) i) $m_{PQ} = \dfrac{y_P - y_Q}{x_P - x_Q}$

$= \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$= \dfrac{a(p-q)(p+q)}{2a(p-q)}$

$= \dfrac{p+q}{2}$

PQ: $y - y_P = m_{PQ}(x - x_P)$

$y - ap^2 = \left(\dfrac{p+q}{2}\right)(x - 2ap)$

$y = \left(\dfrac{p+q}{2}\right)x - \dfrac{2ap^2}{2} - 2apq + ap^2$

$y = \left(\dfrac{p+q}{2}\right)x - apq$

ii) For focal chord, $(0,a)$ lies on PQ.

$\therefore a = \left(\dfrac{p+q}{2}\right)\times 0 - apq$

$a = -apq$

$pq = -1$

(iii) $\angle NPT = 90^\circ$ (NP⊥PT, given)

$\angle NQT = 90^\circ$ (NQ⊥QT, given)

∴ opposite angles of PTQN add to 180°.

∴ PTQN is a cyclic quadrilateral.

(iv) PT has slope p, QT has slope q.

And $pq = -1$.

∴ PT ⊥ QT.

∴ $\angle QTP = 90^\circ$

∴ QP is a diameter of PTQN (angle in a semicircle)

∴ the midpoint of PQ is the centre of C.

Question 12

d) (v) $C = \left(\dfrac{2ap+2aq}{2}, \dfrac{ap^2+aq^2}{2}\right)$

$x = a(p+q) \Rightarrow p+q = \dfrac{x}{a}$

$y = \dfrac{a}{2}(p^2+q^2)$

$= \dfrac{a}{2}\left[(p+q)^2 - 2pq\right]$

$= \dfrac{a}{2}\left((p+q)^2 + 2\right)$ as $pq = -1$.

$= \dfrac{a}{2}\left(\left(\dfrac{x}{a}\right)^2 + 2\right)$

$= \dfrac{a}{2}\left(\dfrac{x^2}{a^2} + 2\right)$

$y = \dfrac{x^2}{2a} + a$ or $2a(y-a) = x^2$

Question 13

a) Prove true for $n=1$

$11^2 + 12^1 = 121 + 12$

$= 133$ which is divisible by 133.

Assume true for $n=k$.

i.e. $11^{k+1} + 12^{2k-1} = 133m$, where m is an integer.

Prove true for $n=k+1$.

∴ R.T.P. $11^{k+2} + 12^{2k+1} = 133n$, where n is an integer.

$11^{k+2} + 12^{2k+1} = 11 \times 11^{k+1} + 144 \times 12^{2k-1}$

$= 11 \times (133m - 12^{2k-1}) + 144 \times 12^{2k-1}$ by the inductive hypothesis

$= 11 \times 133m + 12^{2k-1}(144 - 11)$

$= 133(11m + 12^{2k-1})$

$= 133n$, where n is an integer.

∴ by the principal of mathematical induction, true for $n \ge 1$.

Question 13

(b) (i) $\ddot{x} = x + 3$

$\dfrac{d(\frac{1}{2}v^2)}{dx} = x + 3$

$\frac{1}{2}v^2 = \int x + 3 \, dx$

$= \dfrac{x^2}{2} + 3x + C_1$

$v^2 = x^2 + 6x + C_2$

When $x=0$, $v=3$.

$9 = 0 + 0 + C_2$

$C_2 = 9$

$\therefore v^2 = x^2 + 6x + 9$

$v^2 = (x+3)^2$

$v = \pm(x+3)$

But $v=3$ when $x=0$.

$\therefore v = x + 3$.

(ii) $\dfrac{dx}{dt} = x + 3$

$\dfrac{dt}{dx} = \dfrac{1}{x+3}$

$t = \ln(x+3) + C$

When $t=0$, $x=0$.

$0 = \ln 3 + C$

$\therefore C = -\ln 3$

$t = \ln(x+3) - \ln 3$

$e^t = \dfrac{x+3}{3}$

$x = 3e^t - 3$

Question 13

c) (i) P(correct choice) $= \frac{1}{4}$.

∴ P(15 correct choices) $= \binom{30}{15}\left(\frac{1}{4}\right)^{15}\left(\frac{3}{4}\right)^{15}$

~~= 155117520~~

$= 0.001930545\ldots$

(ii) Let P_k be P(k correct choices).

$\dfrac{P_{k+1}}{P_k} = \dfrac{\binom{30}{k+1}\left(\frac{1}{4}\right)^{k+1}\left(\frac{3}{4}\right)^{29-k}}{\binom{30}{k}\left(\frac{1}{4}\right)^{k}\left(\frac{3}{4}\right)^{30-k}}$

$= \dfrac{\dfrac{30!}{(k+1)!\,(29-k)!} \times \dfrac{1}{4}}{\dfrac{30!}{k!\,(30-k)!} \times \dfrac{3}{4}}$

$= \dfrac{1}{3}\,\dfrac{(30-k)}{k+1}$

$= \dfrac{30-k}{3k+3}$

We need $P_{k+1} > P_k$

$\therefore \dfrac{P_{k+1}}{P_k} > 1$

$\dfrac{30-k}{3k+3} > 1$

$30 - k > 3k + 3$

$27 > 4k$

$k < \dfrac{27}{4}$

$\therefore k = 6$.

$P_7 = \binom{30}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^{23}$

$= 0.166235674$

$\doteqdot 16.6\%$

∴ 16.6% chance he guesses 7 correctly.

Question 13

(d) $\frac{dT}{dt} = -k(T-20)$

(i) $T = 20 + Ae^{-kt}$

$\frac{dT}{dt} = -kAe^{-kt}$

$= -k(Ae^{-kt} + 20 - 20)$

$= -k(T-20)$

(ii) when $t=0$, $T=160$.

$160 = 20 + Ae^{0}$

$\therefore A = 140$

$100 = 20 + 140e^{-10k}$

$80 = 140e^{-10k}$

$\ln\left(\frac{4}{7}\right) = -10k$

$k = \frac{\ln\left(\frac{4}{7}\right)}{-10}$

(iii) $50 = 20 + 140\,e^{\frac{\ln(\frac{4}{7})t}{10}}$

$\ln\left(\frac{3}{14}\right) = \frac{\ln\left(\frac{4}{7}\right)}{10}t$

$t = \frac{10\ln\left(\frac{3}{14}\right)}{\ln\left(\frac{4}{7}\right)}$

$= 27.52683313\ldots$

$\doteq 28$ minutes.

Question 14

a) (i) $x = \frac{2}{\sqrt{3}}\cos 3t + 2\sin 3t$

$\dot{x} = \frac{-6}{\sqrt{3}}\sin 3t + 6\cos 3t$

$\ddot{x} = \frac{-18}{\sqrt{3}}\cos 3t - 18\sin 3t$

$= -9\left(\frac{2}{\sqrt{3}}\cos 3t + 2\sin 3t\right)$

$= -3^2 x$

(ii) $R = \sqrt{a^2+b^2}$ $\qquad \tan\theta = \frac{\frac{2}{\sqrt{3}}}{2}$

$= \sqrt{\frac{4}{3}+4}$ $\qquad = \frac{1}{\sqrt{3}}$

$= \frac{\sqrt{16}}{\sqrt{3}}$ $\qquad \therefore \theta = \frac{\pi}{6}$.

$= \frac{4}{\sqrt{3}}$

$\therefore x = \frac{4}{\sqrt{3}}\sin\left(3t + \frac{\pi}{6}\right)$

$\therefore$ period $= \frac{2\pi}{3}$, amplitude $= \frac{4}{\sqrt{3}}$

b) $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$

Integrating both sides w.r.t. x.

$\frac{(1+x)^{n+1}}{n+1} = \sum_{k=0}^{n}\binom{n}{k}\frac{x^{k+1}}{k+1} + C$

Let $x=0$:

$\frac{1}{n+1} = 0 + C$

$C = \frac{1}{n+1}$

Question 14

(b) continued.

Let $x=1$:

$\frac{2^{n+1}}{n+1} = \sum_{k=0}^{n}\frac{1^{k+1}}{k+1}\binom{n}{k} + \frac{1}{n+1}$

$\frac{2^{n+1}-1}{n+1} = \sum_{k=0}^{n}\frac{1}{k+1}\binom{n}{k}$ ∎

(c)(i) $\ddot{y} = -g$

$\dot{y} = -gt + C$

When $t=0$, $\dot{y} = V\sin\theta$.

$\therefore C = V\sin\theta$.

$y = \frac{-gt^2}{2} + Vt\sin\theta + C$

When $t=0$, $y=h$

$h = 0 + 0 + C$

$\therefore C = h$

$\therefore y = \frac{-gt^2}{2} + Vt\sin\theta + h$.

(ii) $t = \frac{x}{V\cos\theta}$

$y = V\left(\frac{x}{V\cos\theta}\right)\sin\theta - \frac{g}{2}\left(\frac{x^2}{V^2\cos^2\theta}\right) + h$

$= h + x\tan\theta - \frac{g}{2V^2\cos^2\theta}x^2$

(iii) Ball clears net if $y > h$ when $x = 9$.

$h + 9\tan\theta - \frac{81g}{2\cos^2\theta . V^2} > h$

$9\tan\theta > \frac{81g}{2\cos^2\theta . V^2}$

$V^2 > \frac{9g}{2\cos^2\theta\tan\theta}$

$V^2 > \frac{9g}{2\sin\theta\cos\theta}$

Question 14

c) (iv) Ball hits attack line when $x = 12$, $y = 0$ and $V^2 > \frac{9g}{2\sin\theta\cos\theta}$

Now,

$0 = h + 12\tan\theta - \frac{144g}{2V^2\cos^2\theta}$

$\frac{144g}{2V^2\cos^2\theta} = h + 12\tan\theta$

$\frac{144g}{2\cos^2\theta} = (h + 12\tan\theta)V^2$

$\frac{144g}{2\cos^2\theta} > \frac{(h+12\tan\theta)9g}{2\sin\theta\cos\theta}$

$\frac{8}{\cos\theta} > \frac{h+12\tan\theta}{2\sin\theta}$

$16\tan\theta > h + 12\tan\theta$

$4\tan\theta > h$

$\tan\theta > \frac{h}{4}$.

(v) $\tan\theta > \frac{h}{4}$

$\tan\theta > \frac{2.43}{4}$

$\theta > \tan^{-1}\left(\frac{2.43}{4}\right)$

$\theta > 31.28°$